FORMULATION OF STATISTICAL EQUATION OF MOTION OF BLOOD

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ABSTRACT A general form of the statistical equation of motion of the blood is derived by averaging the motion of individual elements of the blood over a small volume in space. This equation can be transformed into an explicit form to find the constitutive equations of the blood provided the detailed motion of the plasma in some neighborhood of a suspended particle is known. As a demonstration of such transformation, the general form of the statistical equation of motion is applied to the suspension of sphere with a very low concentration to find the effective viscosity.

INTRODUCTION

The blood is a particulate medium composed of a suspending liquid (plasma) and the suspended particles (red cells, white cells, platelets, etc.). The motion of the blood in the small blood vessels (arterioles and capillaries) should be described by the motion of plasma and that of the individual particles suspended in it. In the large blood vessels (heart, aorta, and large arteries) the motion of the blood should be described in some statistical manner because the detailed information about the motion of the individual elements of the blood is not only technically unavailable but also practically unnecessary. The blood may be regarded as a continuum if we are concerned with phenomena whose characteristic lengths are much larger than the dimensions of the suspended particles. Since the blood is a suspension and the viscosity of the suspension is a well-known problem, it is useful to mention the following facts about this problem: (*a*) Virtually all existing treatments of the viscosity of the suspension, with a few exceptions, are confined to an evaluation of the additional energy dissipation due to the suspended particles. (*b*) Often the dynamical meaning of the suspension viscosity is not referred to the true state of the motion of the suspension (see reference 4). It may be seen that the energy approach supplies only a part of the information about the constitutive equations of the suspension. A more satisfactory approach is offered by Burgers (1938) who presents a dynamical definition of the viscosity of the suspensions.

STATISTICAL EQUATION OF MOTION OF THE BLOOD

The description of the motion of the blood as a continuum is based on a very simple idea. We can always prepare a system of a pure fluid in such a way that the motion of

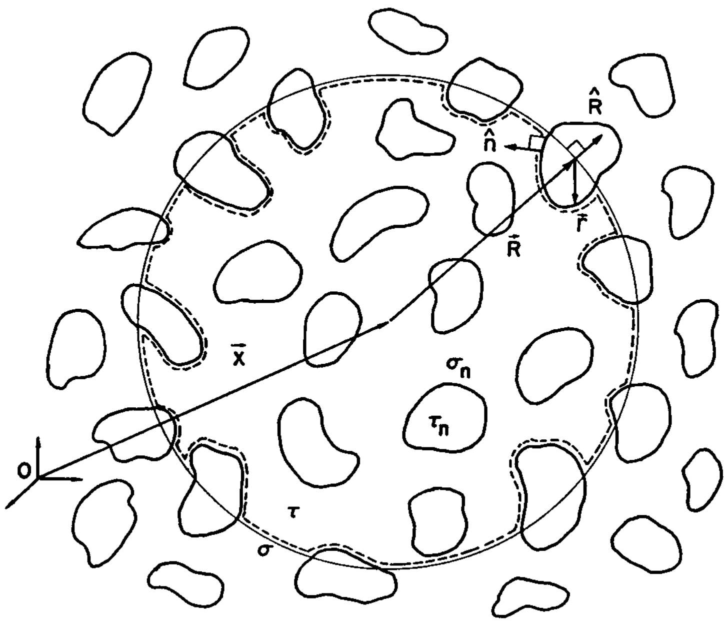

FIGURE 1 The material volume τ with surface σ, over which the quantities are averaged.

the pure fluid and that of the blood look the same in a certain region. We may call such flow system of the pure fluid "the kinematically equivalent system", which literally represents the apparent motion of the blood. The motion of these two systems is only apparently the same because they look the same only if we observe them from a distance which is much greater than the dimensions of the suspended particles. If we observe them at a close distance, we shall find the difference; the velocity of the pure fluid has a smooth distribution in space while that of the blood has fine structure due to the suspended particles. The equations which govern the apparent motion of the blood (the motion of the kinematically equivalent system) can be obtained from the equation of motion of the individual elements of the blood by some averaging process. In doing so, we shall find a problem: how is the apparent motion of the blood related to the true motion of the individual elements of the blood? In this study, we will assume the following proposition; the apparent value is equal to the averaged value over a small volume of the blood in space. The dimensions of the small volume over which the averaged value is taken has to be much less than the characteristic length of the flow system, but it has to be much larger than the characteristic lengths of the suspended particles. Although we assumed that the observed value (apparent value) is the averaged value over a small volume of the blood, it is convenient to introduce another averaged value; the averaged value over a small volume of the plasma. We designate the averaged value of Q over a small blood volume by $\langle Q \rangle$ and that over a small plasma volume by $\overline{Q}$.

Now, returning to the problem of the statistical equation of motion of the blood, we first consider the equation of motion of the plasma contained in a spherical volume of the blood of radius R, whose volume and surface are designated by τ and σ, respectively (see Fig. 1). We choose the spherical volume τ in such a way that its radius R is much smaller than the characteristic length of the system, but much larger than the dimensions of the suspended particles. The blood in the spherical volume τ is composed of the plasma and suspended particles. The volume τ can be split into two parts; the volume occupied by the plasma and the volume occupied by suspended particles. We designate the plasma portion of τ by τ'. The surface σ of the spherical volume τ is a hypothetical surface defined in two different regions; the plasma region and the region inside of the suspended particles. We designate the portion of σ which lies in the plasma region by σ'. Let us define a closed surface which is composed of σ' and the inner portion of the natural surface (plasma-particle interface) of the particles lying on σ. This surface is represented by the dotted line in the Fig. 1, which we designate by s.

The equation of motion of the plasma contained in τ can be written as:

$$\frac{D}{Dt}\int_{\tau'} \rho \mathbf{v}\, d\tau = \mathbf{f}\left(\int_{\tau'} \rho\, d\tau\right) + \oint_{s} \mathbf{t}\, d\sigma + \sum_{n=1} \oint_{\sigma_n} \mathbf{t}\, d\sigma, \tag{1}$$

where ρ is the density of the plasma, $\mathbf{v}$ is the velocity, $\mathbf{f}$ is the body force per unit mass, $\mathbf{t}$ is the stress exerted on the surface of the plasma volume τ' by the surrounding plasma or the suspended particles, D/Dt designates the partial time derivative holding σ fixed, and σ_n designates the surface of the nth particle suspended in τ. The summation over n in (1) is for all particles in τ, excluding those on σ. The equation of motion of the nth particle in τ can be written as,

$$\frac{d}{dt}\int_{\tau_n} \rho_p \mathbf{v}\, d\tau = \mathbf{f}\int_{\tau_n} \rho_p\, d\tau - \oint_{\sigma_n} \mathbf{t}\, d\sigma, \tag{2}$$

where ρ_p is the density of the particle, τ_n designates the volume of the nth particle, and d/dt is the partial time derivative holding σ_n fixed. The stress $\mathbf{t}$ of the rhs. of equation 2 is the same $\mathbf{t}$ as that of equation 1, which is the stress exerted on the plasma by the particle. When equation 2 is substituted into (1) to eliminate the surface integral over σ_n, the following equation results;

$$\frac{D}{Dt}\int_{\tau'} \rho \mathbf{v}\, d\tau + \sum_{n=1} \frac{d}{dt}\int_{\tau_n} \rho_p \mathbf{v}\, d\tau = \mathbf{f}\left(\int_{\tau'} \rho\, d\tau + \sum_{n=1}\int_{\tau_n} \rho_p\, d\tau\right) + \oint_{s} \mathbf{t}\, d\sigma. \tag{3}$$

By dividing (3) by τ, we obtain

$$(1-\varphi)\rho\overline{\frac{D\mathbf{v}}{Dt}} + \varphi\rho_p\left\langle\frac{d\mathbf{v}_p}{dt}\right\rangle = [(1-\varphi)\rho + \varphi\rho_p]\mathbf{f} + \frac{1}{\tau}\oint_{s} \mathbf{t}\, d\sigma, \tag{4}$$

where φ designates the volume concentration of the particle, which is equal to $\varphi = \tau - \tau'/\tau$, and $\mathbf{v}_p$ designates the linear velocity of the suspended particle and

$$\rho \overline{\frac{D\mathbf{v}}{Dt}} = \frac{1}{\tau'} \frac{D}{Dt} \int_{\tau'} \rho \mathbf{v} \, dz,$$

$$\varphi \rho_p \left\langle \frac{d\mathbf{v}_p}{dt} \right\rangle = \frac{1}{\tau} \sum_{n=1} \frac{d}{dt} \int_{\tau_n} \rho_p \mathbf{v} \, d\tau,$$

here ρ and ρ_p are assumed constant. The surface s is composed of the plasma portion of the spherical surface and the particle-plasma interface of the suspended particles lying on σ. If we designate the inner portion of the particle-plasma interface of the nth particle lying on σ by σ_n, we can write equation 4 as follows

$$(1 - \varphi)\rho \overline{\frac{D\mathbf{v}}{Dt}} + \varphi \rho_p \left\langle \frac{d\mathbf{v}_p}{dt} \right\rangle = [(1 - \varphi)\rho + \varphi \rho_p]\mathbf{f}$$

$$+ \frac{1}{\tau} \int_{\sigma'} \mathbf{t} \, d\sigma + \frac{1}{\tau} \sum_{n=1} \int_{\sigma_n} \mathbf{t} \, d\sigma. \quad (5)$$

The plasma is supposedly Newtonian and the stress in the plasma is related to the stress tensor by the following equation;

$$\mathbf{t} = \hat{n} \cdot (-\mathfrak{I} p + 2\mu \mathfrak{D}), \quad (6)$$

where $\hat{n}$ is the unit normal vector of the surface element on which the stress $\mathbf{t}$ is acting, $\mathfrak{I}$ designates the idemfactor, p is the pressure, μ is the shear viscosity of the plasma, and $\mathfrak{D}$ designates the deformation rate tensor. When equation 6 is substituted into equation 5, the following equation can be obtained:

$$(1 - \varphi)\rho \overline{\frac{D\mathbf{v}}{Dt}} + \varphi \rho_p \left\langle \frac{d\mathbf{v}_p}{dt} \right\rangle = [(1 - \varphi)\rho + \varphi \rho_p]\mathbf{f}$$

$$+ \frac{1}{\tau} \int_{\sigma'} \hat{R} \cdot (-\mathfrak{I} p + 2\mu \mathfrak{D}) \, d\sigma + \frac{1}{\tau} \sum_{n=1} \int_{\sigma_n} \mathbf{t} \, d\sigma \quad (7)$$

where $\hat{R}$ designates the unit outward normal vector on σ.

The average value of the pressure, $\bar{p}$ and that of the deformation rate tensor, $\bar{\mathfrak{D}}$ over the plasma volume τ' should be defined as follows:

$$\int_{\sigma'} \hat{R} \bar{p} \, d\sigma = \int_{\sigma'} \hat{R} p \, d\sigma \quad (8)$$

$$\int_{\sigma'} \hat{R} \cdot \bar{\mathfrak{D}}\, d\sigma = \int_{\sigma'} \hat{R} \cdot \mathfrak{D}\, d\sigma. \tag{9}$$

The above equation is not trivial because $\bar{p}$ and $\bar{\mathfrak{D}}$ have continuous derivative in the space, while p and $\mathfrak{D}$ are not continuous in the space. We can show that the above equations uniquely determine the average value over the plasma volume. Let us suppose that there exist two different average value $\bar{p}_1$ and $\bar{p}_2$. By using equation 8 one finds that $\int_{\sigma'} \hat{R}(\bar{p}_1 - \bar{p}_2)\, d\sigma = 0$. Since $\bar{p}_1$ and $\bar{p}_2$ have continuous derivative in the space,

$$\int_{\sigma'} \hat{R}(\bar{p}_1 - \bar{p}_2)\, d\sigma = \frac{\sigma'}{\sigma} \oint_{\sigma} \hat{R}(\bar{p}_1 - \bar{p}_2)\, d\sigma$$

which can be transformed into a volume integral $\int_\tau \nabla(\bar{p}_1 - \bar{p}_2)\, d\tau = 0$. This requires that $\nabla(\bar{p}_1 - \bar{p}_2) = 0$ because τ is not an uniquely defined volume. Thus, we proved that $\bar{p}_1$ differs from $\bar{p}_2$ by a constant, which we can always set equal to zero. Similar argument results $\nabla \cdot (\bar{\mathfrak{D}}_1 - \bar{\mathfrak{D}}_2) = 0$. The only solution of this equation which gives a bounded flow in an unbounded space is $\bar{\mathfrak{D}}_1 = \bar{\mathfrak{D}}_2$. This has to be so for a bounded space because the averaging process over the plasma volume should not depend on the nature of the boundary of the flow system. Therefore, the integrand of the surface integral over σ' in equation 7 can be replaced by its averaged value over a small plasma volume. By doing so, we can write equation 7 in the following form;

$$\begin{aligned}(1 - \varphi)\rho \overline{\frac{D\mathbf{v}}{Dt}} + \varphi\rho_p \left\langle \frac{d\mathbf{v}_p}{dt} \right\rangle &= [(1 - \varphi)\rho + \varphi\rho_p]\mathbf{f} \\ &+ \frac{1}{\tau}\left[\int_{\sigma'} \hat{R} \cdot (-\mathfrak{g}\bar{p} + 2\mu\bar{\mathfrak{D}})\, d\sigma - \sum_{n=1} \int_{\sigma_n} \hat{n} \cdot (-\mathfrak{g}\bar{p} + 2\mu\bar{\mathfrak{D}})\, d\sigma\right] \\ &+ \frac{1}{\tau}\sum_{n=1} \int_{\sigma_n} [\mathbf{t} + \hat{n} \cdot (-\mathfrak{g}\bar{p} + 2\mu\bar{\mathfrak{D}}]\, d\sigma,\end{aligned} \tag{10}$$

where $\hat{n}$ designates the unit outward normal vector on σ_n. We rearranged the rhs of (10) so that the term in the square bracket becomes a closed surface integral.

Finally, we obtain the following form of the statistical equation of motion of the blood from equation 10 after transforming the closed surface integral of the rhs of (10) into a volume integral by using the divergence theorem;

$$(1 - \varphi)\rho \overline{\frac{D\mathbf{v}}{Dt}} + \varphi\rho_p \left\langle \frac{d\mathbf{v}_p}{dt} \right\rangle = [(1 - \varphi)\rho + \varphi\rho_p]\mathbf{f} - \nabla\bar{p} + 2\mu\nabla \cdot \bar{\mathfrak{D}} + \nabla \cdot \mathfrak{Z}_\varphi, \tag{11}$$

where $\mathfrak{Z}_\varphi$ designates the induced stress tensor due to the suspended particles, whose divergence is given by

$$\nabla \cdot \mathfrak{I}_\varphi = \frac{1}{T} \sum_{n=1} \int_{\sigma_n} [\mathbf{t} + \hat{n} \cdot (-\mathfrak{g}\bar{p} + 2\mu\bar{\mathfrak{D}})] \, d\sigma, \tag{12}$$

here **t** designates the stress exerted on the plasma by the suspended particles.

Equation 11 is a general form of the statistical equation of motion of the blood. This equation has to be further transformed into a more explicit form which involves $\langle p \rangle$, $\langle \mathbf{v} \rangle$, and φ as the unknowns. In other words, the inertia force and the induced stress tensor have to be evaluated in terms of the above quantities. In general, the evaluation of these quantities requires the detailed information about the size, shape, and orientation of the suspended particles. The algebra involved in doing so is also complicated except a very simple case. In order to demonstrate how equation 11 can be transformed into an explicit form, we shall work out the problem for the case of the dilute suspension of sphere in the following section.

DILUTE SUSPENSION OF SPHERE

We may regard the suspension of sphere as a simple mathematical model of the blood. Let us first consider the case of very low concentration where the suspended spheres are far apart from each other. If the suspended spheres are very small, the inertia force of the sphere is negligible compared to the surface force exerted by the surrounding fluid. Therefore, the difference in the linear or angular motion between the sphere and the surrounding fluid is due to the nonuniformity of the motion of the fluid in the neighborhood of the sphere. The Laplacian of the velocity is equal to the difference in the velocity between the true value and the averaged value and, consequently, the nonuniformity of the linear and angular velocity in the neighborhood of the sphere is order of $a^2\nabla^2\langle \mathbf{v} \rangle$ and $a^2\nabla^2\langle \nabla\mathbf{v} \rangle$, respectively. Thus, we find that the difference in the linear and angular velocity between the suspended sphere and the surrounding fluid is order of a^2/L^2, where a is the radius of the sphere and L designates the characteristic length of the shearing motion (see reference 6). Therefore, the small suspended particles are more or less frozen in into the suspending plasma and the differences between $\bar{\mathbf{v}}$ and $\langle \mathbf{v}_p \rangle$, and $\overline{D\mathbf{v}}/Dt$ and $\langle d\mathbf{v}_p/dt \rangle$ are negligible as long as a/L is very small. Since the particle moves and rotates with the surrounding plasma, the true motion of the plasma differs from the averaged motion over a small plasma volume due to the perturbed motion caused by the nondeformability of the suspended sphere. If this perturbed motion of the plasma is designated by p' and $\mathbf{v}'$, the adherence condition of the plasma to the surface of the sphere requires that

$$\mathbf{v}' = -a\hat{n} \cdot \bar{\mathfrak{D}} \qquad \text{at the surface of the sphere} \tag{13}$$

where $\hat{n}$ designates the unit outward normal vector on the surface of the sphere. According to equation 13, the order of magnitude of $\mathbf{v}'$ is equal to a/L. Since a/L

is supposedly very small, $\mathbf{v}'$ can be found from the equation of low Reynolds number flow, which is

$$- \nabla p' + \mu \nabla^2 \mathbf{v}' = 0. \tag{14}$$

The solution of equation 14 satisfying the boundary condition equation 13 can be easily found to be (see reference 5, 77)

$$\mathbf{v}' = -\frac{a^5}{r^5}\mathbf{r}\cdot\overline{\mathfrak{D}}(\mathbf{R}) - \frac{5}{2}\left(\frac{a^3}{r^5} - \frac{a^5}{r^7}\right)\mathbf{rr}:\overline{\mathfrak{D}}(\mathbf{R})\mathbf{r}, \tag{15}$$

$$p' = -5\mu\frac{a^3}{r^5}\mathbf{rr}:\overline{\mathfrak{D}}(\mathbf{R}), \tag{16}$$

where $\mathbf{r}$ designates the vector from the center of the sphere $\mathbf{R}$ to the point of observation. From this solution, we find the stress exerted on the plasma by the surface of the sphere as follows:

$$\mathbf{t}' = -3\mu\hat{n}\cdot\overline{\mathfrak{D}}(\mathbf{R}). \tag{17}$$

The total stress $\mathbf{t}$ can be found by adding the stresses due to the unperturbed motion of the plasma to equation 17. By doing so, we find that

$$\mathbf{t} + \hat{n}\cdot(-\mathscr{I}\bar{p} + 2\mu\overline{\mathfrak{D}}) = -3\mu\hat{n}\cdot\overline{\mathfrak{D}}. \tag{18}$$

$\overline{\mathfrak{D}}(\mathbf{R})$ can be expanded into Taylor series about the center of the spherical volume τ to find that

$$\mathbf{t} + \hat{n}\cdot(-\mathscr{I}\bar{p} + 2\mu\overline{\mathfrak{D}}) = -3\mu[\hat{n}\cdot\overline{\mathfrak{D}}(\mathbf{x}) + \hat{n}\mathbf{R}:\nabla\overline{\mathfrak{D}}(\mathbf{x})], \tag{19}$$

where $\mathbf{x}$ designates the center of τ and $\mathbf{R}$ is vector from $\mathbf{x}$ to the center of the suspended sphere. If we designate the angle between $-\mathbf{R}$ and $\hat{n}$ by θ, we find that

$$\hat{n} = -\hat{R}\cos\theta + \hat{R}_{\perp}\sin\theta, \tag{20}$$

where $\hat{R}$ is the unit vector in the direction of $\mathbf{R}$ and $\hat{R}_{\perp}$ is a unit vector perpendicular to $\mathbf{R}$ and $(\mathbf{R}\times\hat{n})$. By using equations 19 and 20 the divergence of the induced stress tensor of equation 12 can be written as follows:

$$\begin{aligned}\nabla\cdot\mathfrak{J}_{\varphi} = \frac{-3\mu}{\tau}\Bigg[&\sum_{n=1}\int_{\sigma_n}(-\hat{R}\cos\theta + \hat{R}_{\perp}\sin\theta)\,d\sigma\cdot\overline{\mathfrak{D}}(\mathbf{x})\\ &+\sum_{n=1}\int_{\sigma_n}(-\hat{R}\mathbf{R}\cos\theta + \hat{R}_{\perp}\mathbf{R}\sin\theta)\,d\sigma:\nabla\overline{\mathfrak{D}}(\mathbf{x})\Bigg].\end{aligned} \tag{21}$$

It can be shown that the terms of equation 21 involving $\hat{R}$ or $\hat{R}_{\perp}$ odd number of times vanish when the summation over all particles lying on σ is carried over. By omitting such terms from equation 21 we find that

$$\nabla \cdot \mathfrak{J}_\varphi = \frac{3\mu}{\tau} \sum_{n=1} \int_{\sigma_n} \hat{R}\mathbf{R} \cos\theta \, d\sigma : \nabla \bar{\mathfrak{D}}(\mathbf{x}). \tag{22}$$

When the discrete distribution of the particles is approximated by a continuously varying number density N, the summation over all particles on σ can be replaced by an integral over the volume of a shell of thickness $2a$, whose middle surface is σ (see Fig. 2). By doing so, we can write equation 22 as follows:

$$\nabla \cdot \mathfrak{J}_\varphi = \frac{3\mu a^2 N}{\frac{4\pi}{3} R_0^3} \left[\oint d\Omega \hat{R}\hat{R} \int_{R=R_0-a}^{R_0+a} dR R^3 \int_{\theta=0}^{\Theta} \sin\theta \cos\theta \, d\theta \int_{\psi=0}^{2\pi} d\psi \right] : \nabla \bar{\mathfrak{D}}(\mathbf{x}) \tag{23}$$

where R_0 designates the radius of the spherical surface σ, Ω designates the solid angle viewed from the center of σ, Θ is the maximum value of θ for a given σ_n, and ψ is the angle measured on a plane perpendicular to $\hat{R}$ (see Fig. 2). By retaining only the highest order term of equation 23, we find that

$$\nabla \cdot \mathfrak{J}_\varphi = 3\pi\mu a^2 N \left[\int_{R=R_0-a}^{R_0+a} (1 - \cos^2\Theta) \, dR \right] \nabla \cdot \bar{\mathfrak{D}}(\mathbf{x}). \tag{24}$$

Since $R - R_0 = a \cos\Theta$, the integral of equation 24 can be easily evaluated to find that

$$\nabla \cdot \mathfrak{J}_\varphi = 3\mu \left(\frac{4\pi}{3} a^3 N \right) \nabla \cdot \bar{\mathfrak{D}}(\mathbf{x}). \tag{25}$$

We notice that $4\pi/3a^3N$ is none other than the volume concentration of the particles. Therefore, equation 25 can be written as;

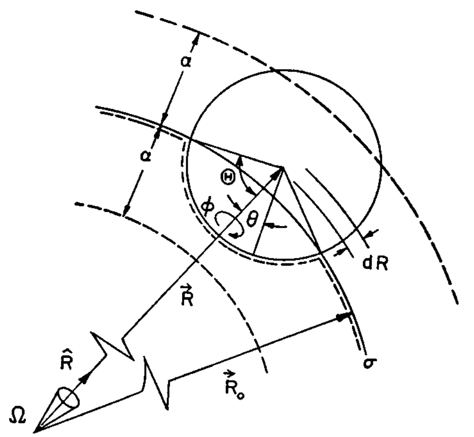

FIGURE 2 Geometry of a particle lying on σ.

$$\nabla \cdot \mathfrak{J}_\varphi = 3\varphi\mu\nabla \cdot \bar{\mathfrak{D}}. \tag{26}$$

Finally, we find the statistical equation of motion of the suspension of sphere by substituting equation 26 into equation 11

$$\langle\rho\rangle \frac{\overline{D\mathbf{v}}}{Dt} = \langle\rho\rangle \mathbf{f} - \nabla\bar{p} + 2\mu\left(1 + \frac{3}{2}\varphi\right)\nabla \cdot \bar{\mathfrak{D}}, \tag{27}$$

where $\langle\rho\rangle$ designates the density of the suspension which is equal to

$$\langle\rho\rangle = (1 - \varphi)\rho + \varphi\rho_p .$$

In obtaining equation 27, we replaced $\langle d\mathbf{v}_p/dt\rangle$ by $\overline{D\mathbf{v}}/Dt$ whose justification has been mentioned at the beginning of this section. We mention once more that equation 27 is valid only for a suspension of sphere with a very low concentration. By using equation 15 it can be shown that the difference between $\overline{D\mathbf{v}}/Dt$ and $D\bar{\mathbf{v}}/Dt$ is order of $\varphi a^2/L^2$, which is negligible. The averaged value over a small plasma volume cannot be observed or measured. It is our proposition that the apparent value (the value represented by the kinematically equivalent system) is equal to the averaged value over a small blood volume. Inside of the suspended sphere, $\mathbf{v}'$ is equal to $-\mathbf{r} \cdot \bar{\mathfrak{D}}$. From this and equation 15 we find that the averaged value of the velocity over a small blood volume differs from that over a small plasma volume by the order of a/L. This statement is also true for the pressure. Therefore, we can replace $\bar{\mathbf{v}}$ and $\bar{p}$ in equation 27 by $\langle\mathbf{v}\rangle$ and $\langle p\rangle$, respectively. The deformation rate tensor vanishes identically inside of the sphere. The ratio of the blood volume over the plasma volume is equal to $(1 - \varphi)$. Therefore, we find that

$$\mathfrak{D} = \frac{1}{1 - \varphi} \langle\mathfrak{D}\rangle.$$

Now, equation 27 can be written in terms of the apparent quantities as follows;

$$\langle\rho\rangle \frac{D\langle\mathbf{v}\rangle}{Dt} = \langle\rho\rangle\mathbf{f} - \nabla\langle p\rangle + \mu \frac{1 + \frac{3}{2}\varphi}{1 - \varphi} \nabla^2\langle\mathbf{v}\rangle, \tag{28}$$

where $\langle\mathbf{v}\rangle$ is the velocity of the kinematically equivalent system whose deformation rate tensor is $\langle\mathfrak{D}\rangle$.

According to equation 28, the relative viscosity of the suspension of sphere is equal to

$$\frac{\mu_e}{\mu} = \frac{1 + \frac{3}{2}\varphi}{1 - \varphi} \tag{29}$$

where μ_e designates the shear viscosity of the suspension. The above equation coincides with Einstein's result (1906) because φ is supposedly very small.

If the volume concentration φ is not very small, we have to solve equation 14 subjected by the boundary conditions that $\mathbf{v}'$ has to vanish on the surface of all spheres in the neighborhood of the source of the perturbation plus the condition of equation 13. Such solution is not available at the present time and the induced stress tensor for a concentrated system cannot be found.

CONCLUSIONS

In general, the method demonstrated in the previous section can be used to transform equation 11 into an explicit form which can be used to determine the motion of blood, if the true motion of the plasma in some neighborhood of a suspended particle is known. Therefore, the problem of the motion of the blood is confined to an accurate evaluation of the perturbed flow of the plasma due to the suspended particles. The constitutive equation of the blood is effected by the form of the induced stress tensor as well as by the inertial force of equation 11. This is so because $\overline{D\mathbf{v}}/Dt$ involves the term $\nabla \cdot \rho\overline{\mathbf{v}'\mathbf{v}'}$ which may not be negligible for a concentrated system. If the suspended particles are deformable, the form of the constitutive equation of the suspension as well as the numerical value involved in it will depend on the property of the suspension as well as the state of the shearing motion, because the orientation and the shape of the particle must depend on the kinematical state of the suspension. It is observed that the increase of the shear viscosity of the blood due to the increase of the hematocrit is not as strong as the case of the suspension of sphere. This is supposedly due to the disc-like shape and the deformation of the red blood cells, whose existence cause much less modification of the shear field of the plasma compared to the rigid sphere. Chen et al. found that the blood viscosity increases markedly as the red cells harden (1967). The theoretical study of the viscosity of a fluid containing small immiscible fluid spheres by Taylor demonstrates the effect of the deformation of suspended particles (1932). It should be mentioned that the equation of continuity and the statistical equation of motion (11) do not determine the motion of the blood completely, because they supply only four equations which involve five unknowns, namely $\langle p \rangle$, $\langle \mathbf{v} \rangle$, and φ. Therefore, the equation governing the distribution of φ has to be derived to make the system of equations complete. Such equation may be derived by considering the motion of the suspended particles as the statistical equation of motion of the blood has been derived by considering the motion of the plasma.

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SYMBOLS

a Radius of the particle.
$\mathfrak{D}$ Deformation rate tensor.
$\mathbf{f}$ Body force per unit mass.
$\mathcal{I}$ Idemfactor.

L Characteristic length of the flow system.
N Number density of the particles.
$\hat{n}$ Unit normal vector.
p Pressure.
$\mathbf{R}$ Vector from the center of the material volume to its surface.
$\hat{R}$ Unit vector along the direction of $\mathbf{R}$.
$\mathbf{r}$ Vector from the center of the particle to the point of observation.
$\mathfrak{J}_{\varphi}$ Induced stress tensor.
$\mathbf{t}$ Stress in the plasma.
t Time.
U Characteristic speed of the flow system.
$\mathbf{v}$ Velocity of the plasma.
$\mathbf{v}_p$ Velocity of the particle.
$\mathbf{x}$ Position vector of the center of the material volume.
μ Shear viscosity of the plasma.
μ_e Shear viscosity of the suspension.
ρ Density of the plasma.
ρ_p Density of the particle.
σ Surface of the material volume.
σ_n Surface of the nth particle.
σ' Plasma portion of σ.
τ Material volume.
τ_n Volume of the nth particle.
τ' Plasma portion of τ.
φ Hematocrit or volume concentration of the particles.
∇ Gradient operator.
∇^2 Laplace operator.
$\cdot$ Dot product.
$:$ Double-dot product.
$\frac{D}{Dt}$ Partial time derivative holding σ fixed.
$\frac{d}{dt}$ Partial time derivative holding σ_n fixed.
$\langle Q \rangle$ Average value of Q over small blood volume.
$\bar{Q}$ Average value of Q over small plasma volume.
Q' $\equiv Q - \langle Q \rangle$.

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